ON THE SET OF STEADY MOTIONS OF A GYROSTAT SATELLITE IN A CENTRAL NEWTONIAN FORCE FIELD AND THE STABILITY OF THESE MOTIONS

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The first four sections of the present paper deal with the problem in restricted formulation, i.e. we assume that the center of mass of the satellite moves as a material point along a circular Keplerian orbit. Assuming that the gyrostatic moment can have arbitrary constant values, we express the set of positions of relative equilibrium of the gyrostat satellite in the orbital coordinate system in a readily understandable geometric form. We then proceed to define the domains of stability and instability.

The results obtained in Sects. 1–4 are evaluated in Sect. 5 from the standpoint of the unrestricted formulation of the problem.

1. We begin by attaching to the satellite the coordinate system $Gx_1x_2x_3$ with its axes directed along the principal central inertial axes.

Let us assume that the satellite consists of a solid hub and symmetric rotors rotating at constant relative angular velocities. The projections on the axes x_i of the moment of relative momenta of the rotors (the gyrostatic moment) are then constant,

$$k_i = \text{const} \qquad (i = 1,2,3) \tag{1.1}$$

In this case the altered potential energy of the gravitational and inertial forces acting on the satellite is given in the orbital coordinate system by the expression (see [1], p. 102)

$$W = {}^3/_2\omega^2(A_1\gamma_1^2 + A_2\gamma_2^2 + A_3\gamma_3^2) - {}^1/_2\omega^2 (A_1\beta_1^2 + A_2\beta_2^2 + A_3\beta_3^2) - \\ - \omega(k_1\beta_1 + k_2\beta_2 + k_3\beta_3)$$

Here ω is the Keplerian orbital angular velocity; A_i are the moments of inertia of the satellite and rotors relative to the axes x_i; γ_i, β_i are the projections on the same axes of the unit vectors $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ of the radius vector of the center of mass G of the satellite and of the normal to the orbital plane, where

$$\psi = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1 \tag{1.2}$$

$$\chi = \gamma_1\beta_1 + \gamma_2\beta_2 + \gamma_3\beta_3 = 0, \qquad \varphi = \beta_1^2 + \beta_2^2 + \beta_3^2 = 1 \tag{1.3}$$

By the Lagrange theorem [2], the relative equilibria of the satellite correspond to the fixed points γ_{i0}, β_{i0} of the function $W(\gamma_i, \beta_i)$ under conditions (1.2), (1.3). We can therefore express the equilibrium equations along with (1.2), (1.3) by way of the Lagrange multipliers λ, μ, ν. We have

$$\partial V / \partial\gamma_i = (\mu + 3A_i\omega)\gamma_i + \lambda\beta_i = 0 \qquad (i = 1,2,3) \tag{1.4}$$

$$\partial V / \partial\beta_i = \lambda\gamma_i + (\nu - A_i\omega)\beta_i - k_i = 0 \qquad (i = 1,2,3) \tag{1.5}$$

$$V = W / \omega + \lambda\chi + {}^1/_2\mu\psi + {}^1/_2\nu\varphi$$

Let us set $\nu = \nu_0$ and $\gamma_i = \gamma_{i0}$ $(i = 1,2,3)$ subject to restriction (1.2) and solve system (1.3)–(1.5) for μ, λ, β_i, k_i.

Multiplying Eqs. (1.4) by γ_i and by $\lambda\beta_i - 3A_i\omega\gamma_i$ and summing over $i = 1,2,3$ with allowance for (1.2), (1.3), respectively, we obtain

$$\mu = \mu_0 = -3\omega \sum_{i=1}^{3} A_i \gamma_{i0}^2, \qquad \lambda = \lambda_0 = \pm 3\omega \left[\sum_{i=1}^{3} A_i^2 \gamma_{i0}^2 - \left(\sum_{i=1}^{3} A_i \gamma_{i0}^2 \right)^2 \right]^{1/2} \quad (1.6)$$

Making use of (1.6) for $\lambda_0 \neq 0$, we infer from (1.4) and (1.5) that

$$\beta_i = \beta_{i0} = \frac{3\omega}{\lambda_0} \left(\sum_{j=1}^{3} A_j \gamma_{j0}^2 - A_i \right) \gamma_{i0} \qquad (i = 1, 2, 3) \quad (1.7)$$

$$k_i = k_{i0} = \lambda_0 \gamma_{i0} + (\nu_0 - A_i \omega) \beta_{i0} \qquad (i = 1, 2, 3) \quad (1.8)$$

Values (1.7) here satisfy Eqs. (1.3).

If $\lambda_0 = 0$, then Eqs. (1.4) are satisfied independently of β_i. This is easy to see by considering the sum of the squares of the left sides of Eqs. (1.4) with allowance for (1.6). We can then take any values satisfying Eqs. (1.3) as our β_{i0} and determine the quantities k_{i0} from (1.5),

$$k_i = k_{i0} = (\nu_0 - A_i \omega) \beta_{i0} \qquad (i = 1, 2, 3)$$

As we see from (1.6)–(1.8), the quantities β_{i0} do not depend on the quantity ν_0 which affects only the choice of the gyrostatic moments k_{i0} and the stability of the equilibria. The quantity ν_0 can be interpreted as the moment of momenta of the satellite relative to the normal to the orbital plane. This can be shown simply by multiplying Eqs. (1.5) by β_i and summing.

The expression in square brackets in (1.6) is readily reducible to the form

$$\sum_{(123)} (A_2 - A_3)^2 \gamma_{20}^2 \gamma_{30}^2$$

which implies that it is nonnegative for all values of γ_{i0} which satisfy (1.2) and vanishes only if one of the principal central inertial axes of the satellite is directed along $\boldsymbol{\gamma}$ in the equilibrium position. Hence, system (1.3)–(1.5) is solvable for μ, λ, β_i, k_i for all values of ν_0 and γ_{i0} related by expression (1.2).

This means that the attitude of the gyrostat satellite relative to the earth in the equilibrium position is arbitrary. Moreover, if $\lambda_0 \neq 0$, i.e. if $\boldsymbol{\gamma}$ does not coincide with a principal central inertial axis of the satellite, then each such direction, or (which is the same thing) each point of the unit sphere (1.2), is associated with two dynamically equivalent equilibrium positions corresponding to different signs of λ_0 and differing by a 180° rotation about $\boldsymbol{\gamma}$; the k_{i0} are of different sign. If $\lambda_0 = 0$, i.e. if a principal central inertial axis of the satellite is directed along $\boldsymbol{\gamma}$ then the equilibrium position is arbitrary (as regards the angle of rotation about $\boldsymbol{\gamma}$).

Let us compare the above set with the two one-parameter families of relative equilibrium positions of a gyrostat satellite investigated in [1],

$$\gamma_1 = 0, \quad \gamma_2 = 0, \quad \gamma_3 = 1, \quad \beta_1 = \sin\theta, \quad \beta_2 = \cos\theta, \quad \beta_3 = 0 \quad (1.9)$$

$$\gamma_1 = 0, \quad \gamma_2 = -\sin\theta, \quad \gamma_3 = \cos\theta, \quad \beta_1 = 0, \quad \beta_2 = \cos\theta, \quad \beta_3 = \sin\theta \quad (1.10)$$

Availing ourselves of the arbitrary choice of the coordinate system $Gx_1x_2x_3$ and of the parameter θ equal to the angle between the x-axis and the normal to the orbital plane, we can make the three following statements.

Families of relative equilibrium positions (1.9), (1.10) correspond in our interpretation to the points of the great circles $\gamma_{i0} = 0$ on unit sphere (1.2).

If the ellipsoid of inertia of the gyrostat satellite is symmetric, then families (1.9) and (1.10) exhaust the entire set of relative equilibrium positions of the gyrostat satellite.

Family (1.9) exhausts all the possible equilibrium positions in which the principal central inertial axes of the satellite are directed along γ, i.e. the case where $\lambda_0 = 0$.

Since the latter case is investigated in detail in [1], we shall assume throughout the remainder of our discussion that $\lambda_0 \neq 0$.

2. The steady motions of the gyrostat satellite considered above are stable by virtue of the Lagrange and Kelvin theorems [2] provided the values of γ_{i0}, β_{i0} minimize the function W under conditions (1.2), (1.3).

Let us consider the three matrices

$$A^{(i)} = \| a_{pq}^{(i)} \| \qquad (a_{pq}^{(i)} = a_{qp}^{(i)};\ p, q = 1, \ldots, 9;\ i = 1, 2, 3)$$

consisting of the second partial derivatives of the function V and differing by virtue of cyclic permutation of the subscripts (123) in the sequence of variables λ, μ, ν, γ_1, γ_2, γ_3, β_1, β_2, β_3 with respect to which the differentiations are carried out. The notation

$$J = A_1\gamma_{10}^2 + A_2\gamma_{20}^2 + A_3\gamma_{30}^2 \tag{2.1}$$

$$a_i = \nu_0 - A_i\omega, \qquad b_i = 3A_i\omega + \mu_0 = 3\omega\,(A_i - J) \quad (i = 1,2,3)$$

enables us to express the nonsymmetric elements $a_{pq}^{(i)}$ with the numbers $p \leqslant q$ in the form

$$\begin{gathered}
a_{17}^{(1)} = a_{24}^{(1)} = \gamma_{10}, \quad a_{18}^{(1)} = a_{25}^{(1)} = \gamma_{20}, \quad a_{19}^{(1)} = a_{26}^{(1)} = \gamma_{30} \\
a_{14}^{(1)} = a_{37}^{(1)} = \beta_{10}, \quad a_{15}^{(1)} = a_{38}^{(1)} = \beta_{20}, \quad a_{16}^{(1)} = a_{39}^{(1)} = \beta_{30} \\
a_{44}^{(1)} = b_1, \quad a_{55}^{(1)} = b_2, \quad a_{66}^{(1)} = b_3, \quad a_{47}^{(1)} = a_{58}^{(1)} = a_{69}^{(1)} = \lambda_0 \\
a_{77}^{(1)} = a_1, \quad a_{88}^{(1)} = a_2, \quad a_{99}^{(1)} = a_3 \qquad (123)
\end{gathered} \tag{2.2}$$

(the cyclic permutation (123) does not extend to the subscripts in $a_{pq}^{(i)}$)

By virtue of (1.3) the quantities β_{10}, β_{20}, β_{30} cannot vanish simultaneously. Let $\beta_{i0} \neq 0$ (i is fixed) in this relative equilibrium position. We can now write the sufficient conditions for the stability of this equilibrium in the form [3] (*)

$$\Delta_7^{(i)} = \beta_{i0}^2 a > 0, \qquad \Delta_8^{(i)} > 0, \qquad \Delta_9^{(i)} = \Delta = a\nu_0^2 + b\nu_0 + c > 0 \tag{2.3}$$

Here the $\Delta_j^{(i)}$ denote the principal diagonal j th order minors with signs opposite to those of the matrix $A^{(i)}$.

Conditions (2.3) are the broadest stability conditions obtainable from the conditional fixed-sign property of the quadratic form corresponding to the second variation.

Kelvin introduced the notions of secular and temporary stability [2]. In this sense conditions (2.3) guarantee secular stability. If conditions (2.3) are violated and the sign of at least one of the inequalities is changed to its opposite, then the equilibrium is unstable in the secular sense.

Expanding the determinant Δ in the first three rows and columns and applying (1.2), (1.3), (1.7), (2.1) and (2.2), we obtain

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*) In proving the conditions [3] for a conditional minimum of the function W we require that the determinant consisting of the first k columns in the Jacobi matrix corresponding to k constraints (1.2), (1.3) be different from zero. However, the theorem remains valid if we assume that the matrix consisting of the first $k+1$ of its columns is of the rank k. This is because the principal diagonal minors $\Delta_j^{(i)}$ beginning with the $(2k+1)$-th order which occur in its formulation are not altered by equivalent permutations of the first $2k+1$ rows and columns in the matrix $A^{(i)}$.

$$\Delta=\sum_{j=1}^{3}b_j\alpha_{j0}^2\sum_{(123)}\beta_{10}^2a_2a_3+\sum_{j=1}^{3}a_j\alpha_{j0}^2\sum_{(123)}\gamma_{10}^2b_2b_3-\lambda_0^2\left(\sum_{j=1}^{3}a_j\gamma_{j0}^2+\sum_{j=1}^{3}b_j\beta_{j0}^2\right) \qquad (2.4)$$

$$\alpha_{10}=\beta_{20}\gamma_{30}-\beta_{30}\gamma_{20}=-3\omega\lambda_0^{-1}(A_2-A_3)\gamma_{20}\gamma_{30} \qquad (123)$$

Recalling expressions (2.1) for a_i, we infer from this that

$$a=\sum_{i=1}^{3}b_i\alpha_{i0}^2,\qquad b=-\omega a\sum_{(123)}\beta_{10}^2(A_2+A_3)+\sum_{(123)}\gamma_{10}^2b_2b_3-\lambda_0^2$$

$$c=\omega^2a\sum_{(123)}\beta_{10}^2A_2A_3-\omega\sum_{i=1}^{3}A_i\alpha_{i0}^2\sum_{(123)}\gamma_{10}^2b_2b_3+\omega\left(4J-3\sum_{i=1}^{3}A_i\beta_{i0}^2\right)\lambda_0^2 \qquad (2.5)$$

By virtue of (1.6), (1.7), (2.1), (2.4) the quantities a, b and c are rational functions of ω, A_i, γ_{i0}.

If β_{10}, β_{20}, $\beta_{30}\neq 0$ simultaneously for some equilibrium position, then, as we infer from [3], conditions (2.3) for $i=1,2,3$ are mutually equivalent. In other words, if conditions (2.3) are fulfilled for some single value of i, then they are fulfilled for any $i=1,2,3$. They are also equivalent to the formally symmetric conditions

$$a>0,\qquad \Delta_8^{(1)}+\Delta_8^{(2)}+\Delta_8^{(3)}=d\Delta\,/\,d\nu_0>0,\qquad \Delta>0 \qquad (2.6)$$

In fact, it is self-evident, that fulfilment of conditions (2.3) (for $i=1,2,3$) implies fulfilment of conditions (2.6). Conversely, fulfilment of conditions (2.6) implies fulfilment of at least one of the inequalities $\Delta_8^{(i)}>0$ $(i=1,2,3)$ (let us say $\Delta_8^{(1)}>0$). Then, clearly, conditions (2.3) are fulfilled for $i=1$, and by virtue of their equivalence, for $i=2,3$ as well.

Let us show that conditions (2.6) are applicable for arbitrary values of β_{i0}, i.e. even when some of the β_{i0} vanish. For example, let $\beta_{10}\neq 0$, and let conditions (2.3) be fulfilled for $i=1$. This means (see [3]) that $\Delta_8^{(i)}\geqslant 0$ $(i=2,3)$, and that conditions (2.6) are fulfilled.

Conversely, let conditions (2.6) be fulfilled for an equilibrium position for which some of the β_{i0} are equal to zero. Under the above assumptions we can always alter the satellite parameters (k_{i0} or γ_{i0}) so slightly and continuously that (a) all the quantities β_{i0} differ from zero, and (b) conditions (2.6) are preserved in continuity in the altered equilibrium position. The altered equilibrium position is then stable (in the secular sense). By virtue of the Poincaré theory of equilibrium bifurcations [3] this means that the initial equilibrium position ($\Delta\neq 0$) is also stable.

Finally let us rewrite conditions (2.6) for the stability of the relative equilibria of the gyrostat satellite in the form

$$a>0,\qquad \nu_0>\nu_2\qquad \left(2a\nu_2=b+\sqrt{b^2-4ac}\right) \qquad (2.7)$$

Here ν_2 is the largest of the roots $\nu_{1,2}$ of the equation $\Delta=0$. The discriminant b^2-4ac of this equation is nonnegative. Otherwise we would have $\Delta\neq 0$ for all ν_0 when $a\neq 0$, and the sign Δ would be the same as that of a. This in turn would imply (by the second condition of (2.6)) that the degree of instability would differ for $\nu_0=\pm M$ (for $a>0$ — 0 and 2, for $a<0$ — 1 and 3; M is a sufficiently large positive number). This contradicts the bifurcation theory [2].

For example, for $\gamma_{10}=\beta_{10}=0$ the determinant Δ breaks down into the rational factors

$$\frac{3\omega\,(A_1 - A_2\gamma_{20}^2 - A_3\gamma_{30}^2)(\nu_0 - A_1\omega) - 9\omega^2\,(A_2 - A_3)^2\gamma_{20}^2\gamma_{30}^2}{\nu_0 - \omega\,(A_2\gamma_{20}^2 + A_3\gamma_{30}^2) - 3\omega\,(A_2 - A_3)\,(\gamma_{20}^2 - \gamma_{30}^2)} \tag{2.8}$$

Conditions (2.7) are equivalent in this case to those obtained by Rumiantsev ([1], p. 108) and are of the form $A_1 - A_2\gamma_{20}^2 - A_3\gamma_{30}^2 > 0, \qquad \nu_0 > \nu_1, \qquad \nu_0 > \nu_2$

where ν_1 and ν_2 are the roots of expression (2.8) linear in ν_0.

3. Without restricting generality we can assume that $A_1 \geqslant A_2 \geqslant A_3$

Expression (2.5) for a can be expanded into the factors

$$a = 27\omega^3\lambda_0^{-2}\,(J - A_1)(J - A_2)(J - A_3)$$

The sign of the quantity a is determined by the sign of the factor $A_2 - J$. The equation $A_2 - J = 0$ defines the two great circles

$$\sqrt{A_2 - A_3}\,\gamma_3 \pm \sqrt{A_1 - A_2}\,\gamma_1 = 0 \tag{3.1}$$

on sphere (1.2).

Great circles (3.1) divide sphere (1.2) into four domains. The quantity a is positive in the domains containing the points $\gamma_3 = \pm 1$ and negative in the domains containing the points $\gamma_1 = \pm 1$; this applies everywhere except at the points $\gamma_3 = \pm 1$ and $\gamma_1 = \pm 1$ themselves, where $a = 0$. For $A_1 = A_2$ the above great circles merge with the great circle $\gamma_3 = 0$, and a is positive everywhere on sphere (1.2) except at this circle and at the points $\gamma_3 = \pm 1$. For $A_2 = A_3$ they merge with the great circle $\gamma_1 = 0$ and a is negative everywhere on the sphere except at this circle and at the points $\gamma_1 = \pm 1$.

According to stability conditions (2.7), in those domains where $a > 0$ the relative equilibrium positions are stable if $\nu_0 > \nu_2$ and unstable in the secular sense if $\nu_0 < \nu_2$. In those domains where $a < 0$, the equilibria are unstable in the secular sense for all ν_0.

Moreover, the satellite can be temporarily stable if the degree of instability is even, which is the case if $\Delta > 0$ [2]. This happens in the domains $a < 0$ for $\nu_1 < \nu_0 < \nu_2$ and in the domains where $a > 0$ for $\nu_0 < \nu_1$.

If the degree of instability is odd (if $\Delta < 0$) the equilibrium is unstable [2]. This is possible in the domains where $a < 0$ for $\nu_0 > \nu_2$ and $\nu_0 < \nu_1$ and in the domains where $a > 0$ for $\nu_1 < \nu_0 < \nu_2$.

4. Let us consider the case where some of the rotors rotate at constant angular velocities relative to the hub and the remaining rotors (or all of them) rotate about their own axes, i.e. where the forces acting on these rotors do not produce moments relative to their axes, so that the projections of the absolute angular velocities of the rotors on their axes remain constant. This implies the constancy of the projections of the gyrostatic moment vector on the axes x_i; the gyrostatic moment is equal to the sum of the moments of momenta of the relative (i.e. relative to the hub of the satellite) motions of the rotors which rotate at constant regular velocities and of the axial moments of absolute momenta of the free rotors,

$$k_i^* = \text{const} \qquad (i = 1,2,3) \tag{4.1}$$

All of the rotors in steady motions rotate at constant angular velocities, and

$$k_i^* = k_i + \omega \sum_s J_s l_{si}\,(l_{s1}\beta_1 + l_{s2}\beta_2 + l_{s3}\beta_3) \quad (i = 1, 2, 3) \tag{4.2}$$

where the k_i have the same meaning as in Sect. 1, i.e. they are equal to the projections of the vector of the sum of moments of relative momenta in the given steady motion;

J_s, l_{s1}, l_{s2}, l_{s3} are the axial moments of inertia and the direction cosines of the free-rotor axes.

The potential energy of the reduced system obtained by the elimination of the cyclical coordinates corresponding to the free rotors takes the following form in the case (4.1) (see [1], p. 109):

$$W^* = {}^3/_2\omega^2 (A_1\gamma_1^2 + A_2\gamma_2^2 + A_3\gamma_3^2) - {}^1/_2\omega^2 (A_1\beta_1^2 + A_2\beta_2^2 + A_3\beta_3^2) - \omega (k_1^*\beta_1 + k_2^*\beta_2 + k_3^*\beta_3) + {}^1/_2\omega^2 \sum_s J_s (l_{s1}\beta_1 + l_{s2}\beta_2 + l_{s3}\beta_3)^2$$

By making use of the relations (4.2) we can readily show that $\delta W^* = \delta W$. This means that the sets of steady motions in cases (1.1) and (4.1) coincide [1].

The second variation of the function W^* differs from the second variation of W by a positive term,

$$\delta^2 W^* = \delta^2 W + \omega^2 \sum_s J_s (l_{s1}\delta\beta_1 + l_{s2}\delta\beta_2 + l_{s3}\delta\beta_3)^2$$

This means that if conditions (2.7) of the condition of positive definiteness of $\delta^2 W$ are fulfilled, then the function W^* also has a conditional minimum, and (by the Routh theorem [4]) the steady motions of the gyrostat satellite are also stable provided conditions (2.7) are fulfilled in case (4.1). This was proved in [5] for the case of a single rotor.

However, the conditions of a conditional minimum of the function W^* can also be obtained directly. Exactly as in Sect. 2 they can be reduced to the form

$$a > 0, \qquad \nu_0 > \nu_2^* \qquad (2a\nu_2^* = b^* + \sqrt{b^{*2} - 4ac^*}) \tag{4.3}$$

where

$$b^* = b + a\omega \sum_s J_s [1 - (l_{s1}\beta_{10} + l_{s2}\beta_{20} + l_{s3}\beta_{30})^2]$$

$$c^* = c - \omega\lambda_0^2 \sum_s J_s \Big(\sum_{i=1}^3 l_{si}\gamma_{i0}\Big)^2 - 2\lambda_0\omega \Big(\sum_{i=1}^3 b_i\alpha_{i0}\beta_{i0}\Big) \sum_s J_s \Big(\sum_{i=1}^3 l_{si}\alpha_{i0}\Big)\Big(\sum_{i=1}^3 l_{si}\gamma_{i0}\Big) +$$
$$+ \omega^2 a \sum_{s<r} J_s J_r \Big[\sum_{(123)} \beta_{10} (l_{s2} l_{r3} - l_{r2} l_{s3})\Big]^2 + \omega \Big(\sum_{(123)} \gamma_{10}^2 b_2 b_3\Big) \sum_s J_s \Big(\sum_{i=1}^3 l_{si}\alpha_{i0}\Big)^2 -$$
$$- \omega^2 a \sum_s J_s \sum_{(123)} A_1 (l_{s2}\beta_{30} - l_{s3}\beta_{20})^2$$

Conditions (4.3) are broader than conditions (2.7), so that for $a > 0$ we must have $\nu_2^* \leqslant \nu_2$ (similarly, for $a < 0$ we must have $\nu_1^* \leqslant \nu_1$). The form of conditions (4.3) implies that all of the conclusions of Sect. 3 remain valid in case (4.1) provided we replace ν_1, ν_2 by ν_1^*, ν_2^*.

5. Now let us consider an unrestricted formulation of the problem. We begin by introducing the fixed coordinate system $O\xi_1\xi_2\xi_3$ with its origin at the attracting center. We also attach the coordinate system $Gy_1y_2y_3$ in addition to the system $Gx_1x_2x_3$ (Sect. 1) at the center of mass G of the satellite. The axis y_3 is directed along OG, the axis y_1 is parallel to the plane $O\xi_3\xi_1$ and is directed in the direction of motion. All of the systems are right and rectangular. The position of the satellite hub in the coordinate system $O\xi_1\xi_2\xi_3$ is defined by the spherical coordinates R, $\varkappa$, σ of the center of mass G of the satellite,

$$\xi_1 = R\cos\varkappa \sin\sigma, \qquad \xi_2 = R\sin\varkappa, \qquad \xi_3 = R\cos\varkappa \cos\sigma$$

and by the cosines γ_i, β_i $(i = 1,2,3)$ of the angles between the axes x_i and the axes y_3 and ξ_2, respectively. The quantities γ_i, β_i and $\varkappa$ are related by expression (1.2) and the relations

$$\chi^\circ = \gamma_1\beta_1 + \gamma_2\beta_2 + \gamma_3\beta_3 - \sin\varkappa = 0, \qquad \varphi = \beta_1^2 + \beta_2^2 + \beta_3^2 = 1 \qquad (5.1)$$

The altered potential energy which results if we ignore the cyclical coordinate σ in case (1.1) turns out to be of the form ([1], p. 125)

$$W^\circ(\gamma_i, \beta_i, \varkappa, R) = {}^1/_2 K^2/S - U$$
$$K = k - k_1\beta_1 - k_2\beta_2 - k_3\beta_3, \qquad S = MR^2\cos^2\varkappa + A_1\beta_1^2 + A_2\beta_2^2 + A_3\beta_3^2$$
$$U = fMR^{-1} - {}^3/_2 fR^{-3}[A_1\gamma_1^2 + A_2\gamma_2^2 + A_3\gamma_3^2 - {}^1/_3(A_1+A_2+A_3)]$$

Here U is a function of the gravitational forces, f is the gravitational constant, M is the mass of the satellite, k is the constant moment of momenta of the satellite relative to the axis $O\xi_2$, and the remaining symbols have the same meaning as in Sect. 1.

Let us introduce the function $V^\circ = W^\circ / \omega + \lambda\chi^\circ + {}^1/_2\mu\psi + {}^1/_2\nu\varphi$

where λ, μ, ν are undetermined Lagrange multipliers and ω is some arbitrary constant which we shall interpret (see (5.3)) as the Keplerian circular angular velocity; $\omega^2 = f(R^\circ)^{-3}$. By the Routh theorem for determining steady motions, in addition to (1.2), (5.1) we also have the equations

$$\partial V^\circ / \partial\gamma_i = (3fR^{-3}\omega^{-1}A_i + \mu)\gamma_i + \lambda\beta_i = 0 \qquad (i = 1,2,3)$$
$$\partial V^\circ / \partial\beta_i = \lambda\gamma_i + (\nu - \Omega^2\omega^{-1}A_i)\beta_i - \Omega\omega^{-1}k_i = 0 \qquad (i = 1,2,3) \qquad (5.2)$$
$$\partial V^\circ / \partial\varkappa = \Omega^2 MR^2\sin\varkappa\cos\varkappa - \lambda\cos\varkappa = 0$$
$$\partial V^\circ / \partial R = -\Omega^2 MR\cos^2\varkappa + fMR^{-2} - {}^9/_2 fR^{-4}[A_1\gamma_1^2 + A_2\gamma_2^2 + A_3\gamma_3^2 - {}^1/_3(A_1 + A_2 + A_3)] = 0$$

($\Omega = K/S$ is the true orbital angular velocity of the satellite; it is related to the quantity k).

For fixed values $\nu = \nu_0$, $\gamma_i = \gamma_{i0}$ $(i = 1,2,3)$ related by expression (1.2), system (5.1), (5.2) has solutions of the form

$$\lambda = \lambda^\circ, \mu = \mu^\circ, \beta_i = \beta_i^\circ, k_i = k_i^\circ, \Omega = \Omega^\circ, \varkappa = \varkappa^\circ, R = R^\circ = f^{-1/3}\omega^{-2/3} \qquad (5.3)$$

As in Sect. 1 we obtain

$$\lambda^{\circ 2}\cos^2\varkappa^\circ = 9\omega^2\left(\sum_{i=1}^{3} A_i^2\gamma_{i0}^2 - J^2\right), \qquad \mu^\circ = -3\omega J - \lambda^\circ\sin\varkappa^\circ \qquad (5.4)$$

and for $\lambda^\circ \neq 0$ (the case $\lambda^\circ = 0$ corresponds to the case $\lambda_0 = 0$ and is considered in [1],

$$\beta^\circ_i = -(3\omega A_i + \mu^\circ)(\lambda^\circ)^{-1}\gamma_{i0}, \qquad k_i^\circ = \omega(\Omega^\circ)^{-1}[\lambda^\circ\gamma_{i0} + (\nu_0 - \Omega^{\circ 2}\omega^{-1}A_i)\beta_i^\circ] \quad (i = 1,2,3) \qquad (5.5)$$

For convenient comparison of the orders of smallness of the various quantities we assume that they have all been divided by such quantities as M, ω, $A_1 + A_2 + A_3$ and are therefore dimensionless. The symbols ε and ε^2 denote various quantities of the orders l/R and l^2/R^2 (l is the characteristic dimension of the satellite; ε^2 is not necessarily a positive quantity).

The last two equations of (5.2) with allowance for (5.4), (5.5) define the quantities $\varkappa^\circ$ and Ω°; here $\varkappa^\circ \neq 0$ for $\lambda^\circ \neq 0$, and

$$\varkappa^\circ = \varepsilon^2 \neq 0, \qquad \Omega^\circ = \omega + \varepsilon^2 \qquad (5.6)$$

Expressions (5.4) and (5.5) with allowance for (5.6) now yield

$$\lambda^\circ = \lambda_0 (1 + \varepsilon^2), \quad \mu^\circ = \mu_0 + \lambda_0\varepsilon^2, \quad \beta_i^\circ = \beta_{i0} + \varepsilon^2, \quad k_i^\circ = k_{i0} + (1 + \nu_0)\varepsilon^2 \quad (i = 1,2,3) \tag{5.7}$$

We infer from (5.6) and (5.7) that the steady motions of the gyrostat satellite in case (1.1) in the unrestricted formulation of the problem differ from the positions of the relative equilibria considered in Sect. 1 by quantities of the order of ε^2, which is very small for real earth satellites.

Since $\varkappa^\circ \neq 0$ for $\lambda^\circ \neq 0$, it follows that in none of the steady motions considered here for $\lambda^\circ \neq 0$ does the orbital plane of the center of mass of the satellite pass through the attracting center (such motions in the case $\gamma_{10} = 0$ are investigated in more detail in [6]). The displacement of the orbital plane is equal to $R^\circ \sin\varkappa^\circ$ and is of the order εl.

Thus, steady motions [1] of a gyrostat satellite in which one of its principal central inertial axes is directed along $\boldsymbol{\gamma}$ (the case $\lambda^\circ = \lambda_0 = 0$) while the center of the circular orbit coincides with the attracting center O are in this case exceptional.

We can investigate stability by considering, as in Sect. 2, the three matrices

$$A^{(i\varepsilon)} = \| a_{pq}^{(i\varepsilon)} \| \quad (a_{pq}^{(i\varepsilon)} = a_{qp}^{(i\varepsilon)}; \ p, q = 1, \ldots, 11; \quad i = 1, 2, 3)$$

of the second partial derivatives of the function V° with respect to the variables

$$\lambda, \mu, \nu, \gamma_1, \gamma_2, \gamma_3, \beta_1, \beta_2, \beta_3, \varkappa, R \qquad (123)$$

For the derivatives of the function $\Omega = K / S$ we readily obtain the relations

$$\partial\Omega / \partial\gamma_i = 0, \quad \partial\Omega / \partial\beta_i = \varepsilon^2, \quad \partial\Omega / \partial\varkappa = \varepsilon^2, \quad \partial\Omega / \partial R = \varepsilon \tag{5.8}$$

By virtue of (5.2), (5.6), (5.7), (5.8) the elements $a_{pq}^{(i\varepsilon)}$ with the numbers $p \leqslant q \leqslant 9$ differ only slightly from values (2.3); the nonzero elements turn out to be

$$\begin{gathered}
a_{1,}^{(1\varepsilon)} = a_{24}^{(1\varepsilon)} = \gamma_{10}, \quad a_{18}^{(1\varepsilon)} = a_{25}^{(1\varepsilon)} = \gamma_{20}, \quad a_{19}^{(1\varepsilon)} = a_{26}^{(1\varepsilon)} = \gamma_{30} \\
a_{14}^{(1\varepsilon)} = a_{3,}^{(1\varepsilon)} = \beta_1^\circ, \quad a_{15}^{(1\varepsilon)} = a_{38}^{(1\varepsilon)} = \beta_2^\circ, \quad a_{16}^{(1\varepsilon)} = a_{39}^{(1\varepsilon)} = \beta_3^\circ \\
a_{44}^{(1\varepsilon)} = b_1 + \varepsilon^2, a_{55}^{(1\varepsilon)} = b_2 + \varepsilon^2, a_{66}^{(1\varepsilon)} = b_3 + \varepsilon^2, a_{47}^{(1\varepsilon)} = a_{58}^{(1\varepsilon)} = {}_{69}^{(1\varepsilon)} = \lambda^\circ \\
a_{77}^{(1\varepsilon)} = a_1 + (1 + \nu_0)^2 \varepsilon^2, \ a_{88}^{(1\varepsilon)} = a_2 + (1 + \nu_0)^2\varepsilon^2, \ a_{99}^{(1\varepsilon)} = a_3 + (1 + \nu_0)^2\varepsilon^2 \\
a_{78}^{(1\varepsilon)}, \ a_{.9}^{(1\varepsilon)}, \ a_{89}^{(1\varepsilon)} \sim (1 + \nu_0)^2\varepsilon^2 \qquad (123)
\end{gathered} \tag{5.9}$$

The nonzero elements of the last two columns of the matrix $A^{(i\varepsilon)}$ are of the form

$$\begin{gathered}
a_{10\,10}^{(i\varepsilon)} = M\omega^2 R^{\circ 2} (1 + \varepsilon^2), \quad a_{11\,11}^{(i\varepsilon)} = M\omega^2 (1 + \varepsilon^2) \\
a_{1\,10}^{(i\varepsilon)} = -\cos\varkappa^\circ; \quad a_{7\,10}^{(i\varepsilon)}, \ a_{8\,10}^{(i\varepsilon)}, \ a_{9\,10}^{(i\varepsilon)} \sim (1 + \nu_0)^2\varepsilon^2 \\
a_{3\,11}^{(i\varepsilon)}, a_{4\,11}^{(i\varepsilon)}, a_{5\,11}^{(i\varepsilon)}, a_{6\,11}^{(i\varepsilon)}, a_{10\,11}^{(i\varepsilon)} \sim \varepsilon; \quad a_{7\,11}^{(i\varepsilon)}, a_{8\,11}^{(i\varepsilon)}, a_{9\,11}^{(i\varepsilon)} \sim (1 + \nu_0)\varepsilon
\end{gathered} \tag{5.10}$$

From (5.9), (5.10) we readily obtain the following relations for the principal diagonal minors with signs opposite to those of the matrices $A^{(i\varepsilon)}$:

$$\begin{gathered}
D_7^{(i)} = \Delta_7^{(i)} + \varepsilon^2, \quad D_8^{(i)} = \Delta_8^{(i)} + (1 + \nu_0)\varepsilon^2, \quad D_9^{(i)} = \Delta_9^{(i)} + (1 + \nu_0)^2\varepsilon^2 \\
D_{10}^{(i)} = M\omega^2 R^{\circ 2} [\Delta_9^{(i)} + (1 + \nu_0)^2\varepsilon^2], \quad D_{11}^{(i)} = M^2\omega^4 R^{\circ 2} [\Delta_9^{(i)} + (1 + \nu_0)^2\varepsilon^2]
\end{gathered}$$

By virtue of the Routh theorem [4], we can apply the conditional minimum criterion [3] for $\beta_i^\circ \neq 0$ (i is fixed) to express the stability condition in the form

$$D_7^{(i)} > 0, \ldots, D_{11}^{(i)} > 0$$

or

$$\Delta_7^{(i)} + \varepsilon^2 > 0, \quad \Delta_8^{(i)} + (1 + \nu_0)\varepsilon^2 > 0, \quad \Delta_9^{(i)} + (1 + \nu_0)^2\varepsilon^2 > 0 \tag{5.11}$$

As in Sect. 2, conditions (5.11) can be reduced to the form

$$a + \varepsilon^2 > 0, \quad \nu_0 > \nu_2^\circ \quad (2a\nu_2^\circ = b + \sqrt{b^2 - 4ac} + (1 + \nu_0)\varepsilon^2)$$

A similar situation obtains in case (4.1).

Thus, to within terms of the order ε^2 the geometric interpretation of the set of steady motions given in Sects. 1 and 3 and also stability conditions (2.7) and (4.3) for cases (1.1) and (4.1), respectively, are also valid for the unrestricted formulation of the problem. Conditions (2.7) and (4.3) are the stability conditions with respect to γ_i, β_i, R, $\varkappa$, $\gamma_i^\cdot$, $\beta_i^\cdot$, $R^\cdot$, $\varkappa^\cdot$, $\sigma^\cdot$ with allowance for the perturbability of the orbit.

6. Stability conditions (2.7) and (4.3) remain valid in the case where the satellite contains, in addition to the rotors, cavities completely filled with liquid [7].

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DYNAMICS OF AN ELECTROMAGNETIC TRIGGER REGULATOR WITH TWO PULSES PER PERIOD

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The dynamics of an electromagnetically driven electromechanical trigger regulator with two pulses per period is considered. The nonlinear third-order differential equation is investigated by the method of point transformations. The decomposition of the parameter space into domains whose points correspond to various qualitative structures of the phase space is established. The domains of existence of several stable periodic motions in the parameter space are isolated.